1-10-2013

Diophantine Geometry, Eulerian Number Theory, and Undergraduates

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Christopher Goff
University of the Pacific
San Diego JMM 2013
History of Geometry Session
Thursday, January 10, 3pm

How this talk began…

- JMM 2012, Boston: Mini-course to translate Latin
- I signed up to translate a paper from the Euler Archive: E763.

[Disclaimer: I am not a math historian.]

DE TRIBUS

PLURIBUSVE NUMERIS INVENIENDIS,

QUORUM SUMMA SIT QUADRATUM,

QUADRATORUM VERO SUMMA BIQUADRATUM.

AUCTORE

L. EULERO.

How Euler's paper began…

- "Famous is the problem once proposed by Fermat, and recently studied by … Lagrange …, in which are sought two positive whole numbers, the sum of which is a square, while the sum of the squares is a fourth power."

$$x + y = A^2; \quad x^2 + y^2 = B^4$$

Euler's Solution (part 1)

- Repeated use of identities, such as:

$$x = a^2 - b^2; y = 2ab \Rightarrow x^2 + y^2 = (a^2 + b^2)^2$$

$$a = p^2 - q^2; b = 2pq \Rightarrow x^2 + y^2 = (p^2 + q^2)^4$$

- Algebraic manipulation, kind of like perturbation theory, combined with guess-and-check.
- Finds that $p = 3$ and $q = 2$ is a solution, but that gives $x = -119$. So let $q = 2$ and $p = 3+v$. Then

$$x + y = 1 + 148v + 102v^2 + 20v^3 + v^4$$

Euler's Solution (part 2)

- Guess that $1 + 148v + 102v^2 + 20v^3 + v^4$ is the square of $1+74v - v^2$.

$$(1 + 74v - v^2)^2 = 1 + 148v + 5474v^2 - 148v^3 + v^4$$

- Then $5372v^2 = 168v^3$, whence $v = 1343/42$, and $p = 1469/42$.
- So to clear denominators, let $p = 1469$ and $q = 84$. Then find a and b, and then x and y.

Euler's Answer

"…which, though they exceed a billion, are nevertheless the smallest satisfying the problem:

$$x = 4{,}565{,}486{,}027{,}761$$

$$y = 1{,}061{,}652{,}293{,}520$$

which are the same that Fermat, and others after him, found. The sum of them is the square of the number 2,372,159, while the sum of the squares is the fourth power of the number 2,165,017."

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But…

- What about Lagrange’s and Fermat’s solutions?
- Where to look?
  - École Polytechnique archives
  - Bibliothèque Nationale
  - Internet Archive
- Fortunately, I had some very specific numbers to look for.

Joseph Louis Lagrange (1736 – 1813)

DE LAGRANGE,

PUBLIÉES PAR LES SOINS

DE M. J.-A. SERRET,

SOUS LES AUSPICES

DE SON EXCELLENCE

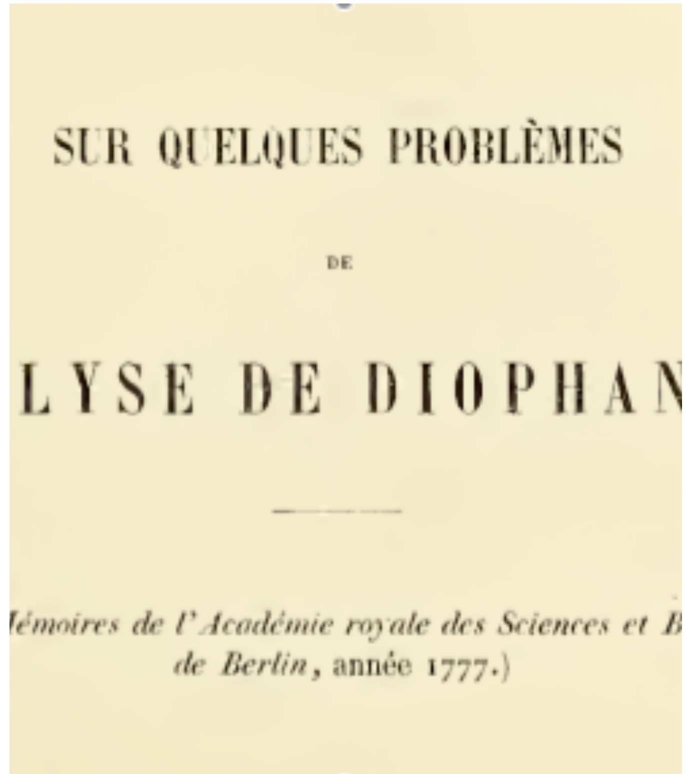
LE MINISTRE DE L'INSTRUCTION PUBLIQUE.

TOME QUATRIÈME.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DE L'ÉCOLE IMPÉRIALE POLYTECHNIQUE, DU BUREAU DES LONGITUDES,



SUR QUELQUES PROBLÈMES

DE

LYSE DE DIOPHAN

lémoires de l'Académie royale des Sciences et B
de Berlin, année 1777.)

Lagrange's Solution (part 1: a different problem)

- Fermat's Method of Descent can show a difference of fourth powers is never a square.
- But the difference of twice a fourth power and a fourth power being a square also "descends," but has a basic solution.

$$2x^4 - y^4 = z^2; \quad x = y = z = 1.$$

- From this initial solution, others can be built by "ascension."
- Lagrange finds: x = 13, y = 1; x = 2,165,017, y = 2,372,159.

On a trouvé aussi pour la première égalité ces autres valeurs

$$x = 13,\ y = 1, \quad \text{et} \quad x = 2165017,\ y = 2372159,$$

Lagrange's Solution (part 2: connections)

- Lagrange ties the previous equation to a Diophantine geometry problem, also considered by Fermat: to find a right triangle whose legs add up to a perfect square, and whose hypotenuse is also a perfect square.

$$p + q = y^2; \quad p^2 + q^2 = x^4$$

- Lagrange noticed that if *p* and *q* satisfy the above equations, then:

$$2x^4 - y^4 = 2(p^2 + q^2) - (p + q)^2 = (p - q)^2 = z^2$$

- Conversely, if you know *y* and *z* above, then:

$$p = \frac{y^2 + z}{2}; \quad q = \frac{y^2 - z}{2}$$

Lagrange’s Answer

From solutions to the first equation, Lagrange finds:

p = 1 and *q* = 0;

p = 120 and *q* = –119;

p = 1,061,652,293,520 and *q* = 4,565,486,027,761

as solutions to the right triangle problem.

Pierre de Fermat (1601 - 1665)

ŒUVRES

DE FERMAT

PUBLIÉES PAR LES SOINS DE

MM. PAUL TANNERY ET CHARLES HENRY

SOUS LES AUSPICES

DU MINISTÈRE DE L'INSTRUCTION PUBLIQUE.

TOME PREMIER.

ŒUVRES MATHÉMATIQUES DIVERSES. — OBSERVATIONS SUR DIOPHANTE.

PARIS,

GAUTHIER-VILLARS ET FILS, IMPRIMEURS-LIBRAIRES

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

Quai des Grands-Augustins, 55.

M DCCC XCI

XLIV (p. 333).

(**Ad idem commentarium.**)

licatis æqualitatibus tractatui multa
eteres nec novi detexerunt. Sufficit n
em et usum asseramus, ut quæstionem
i est, resolvamus.

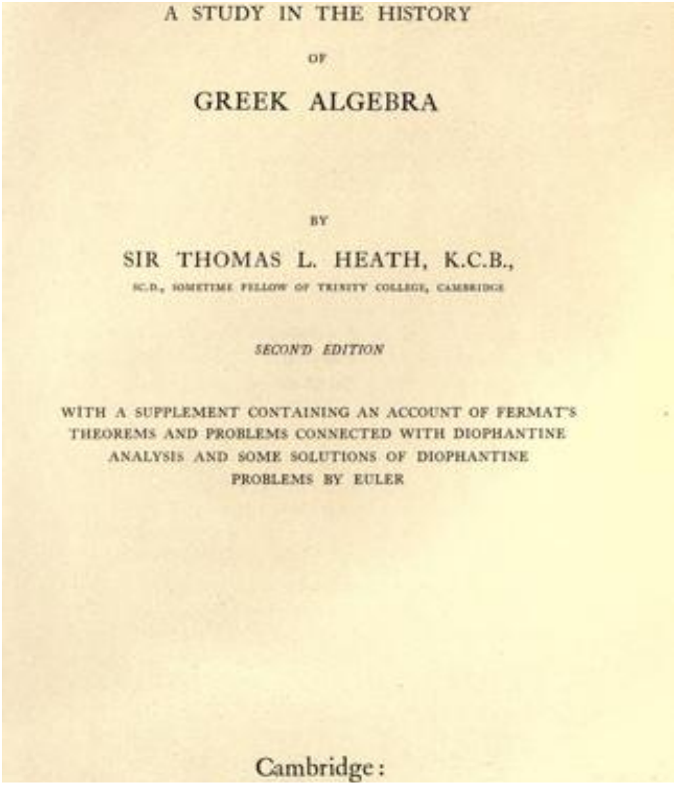
ngulum rectangulum numero, cujus hy
r summa laterum circa rectum (¹).

quæsitum repræsentant tres numeri se
298 610 289, 4 565 486 027 761, 1 061 652 2

Sir Thomas Heath (1861 - 1940)

- *Diophantus of Alexandria*, Cambridge University Press, 1885.

(2nd edition 1910, pp. 297-300)


A STUDY IN THE HISTORY

OF

GREEK ALGEBRA

BY

SIR THOMAS L. HEATH, K.C.B.,

SC.D., SOMETIME FELLOW OF TRINITY COLLEGE, CAMBRIDGE

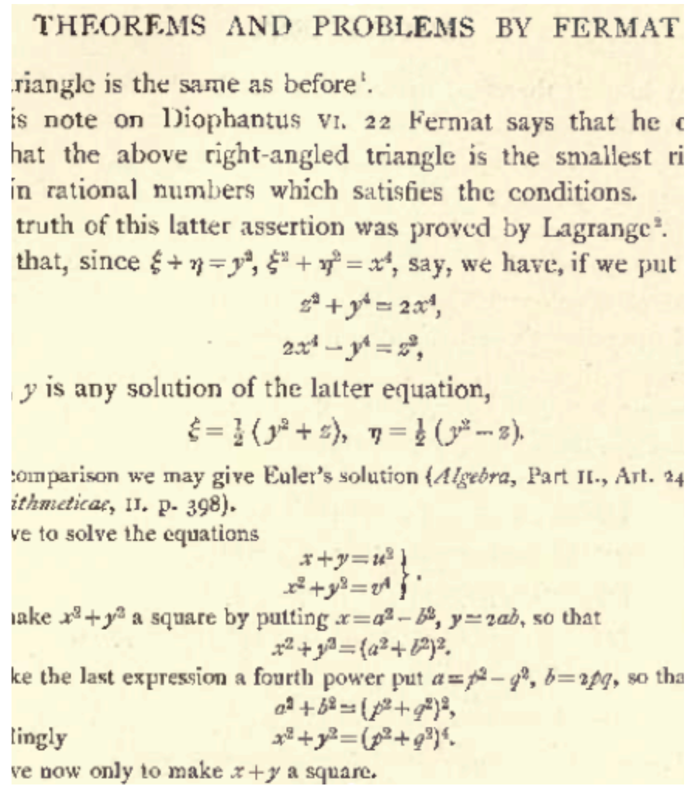
SECOND EDITION

WITH A SUPPLEMENT CONTAINING AN ACCOUNT OF FERMAT'S THEOREMS AND PROBLEMS CONNECTED WITH DIOPHANTINE ANALYSIS AND SOME SOLUTIONS OF DIOPHANTINE PROBLEMS BY EULER

Cambridge:

Heath's Explanation

- Gives a solution.
- Explains Lagrange's justification of the minimality of Fermat's solution.
- Gives Euler's solution as a footnote.
- (Gives essentially an annotated translation of Euler's solution.)



THEOREMS AND PROBLEMS BY FERMAT

riangle is the same as before[1].

is note on Diophantus VI. 22 Fermat says that he c
hat the above right-angled triangle is the smallest ri
in rational numbers which satisfies the conditions.
truth of this latter assertion was proved by Lagrange[2].
that, since $\xi + \eta = y^2$, $\xi^2 + \eta^2 = x^4$, say, we have, if we put

$$z^2 + y^4 = 2x^4,$$

$$2x^4 - y^4 = z^2,$$

y is any solution of the latter equation,

$$\xi = \tfrac{1}{2}(y^2 + z), \quad \eta = \tfrac{1}{2}(y^2 - z).$$

omparison we may give Euler's solution (*Algebra*, Part II., Art. 24
ithmeticae, II. p. 398).
ve to solve the equations

$$\left.\begin{aligned} x+y&=u^2 \\ x^2+y^2&=v^4 \end{aligned}\right\}.$$

ake x^2+y^2 a square by putting $x=a^2-b^2$, $y=2ab$, so that

$$x^2+y^2=(a^2+b^2)^2.$$

ke the last expression a fourth power put $a=p^2-q^2$, $b=2pq$, so tha

$$a^2+b^2=(p^2+q^2)^2,$$

lingly $\quad x^2+y^2=(p^2+q^2)^4.$

ve now only to make $x+y$ a square.

Back to Euler

- Euler did not stop with two numbers; he found three numbers:

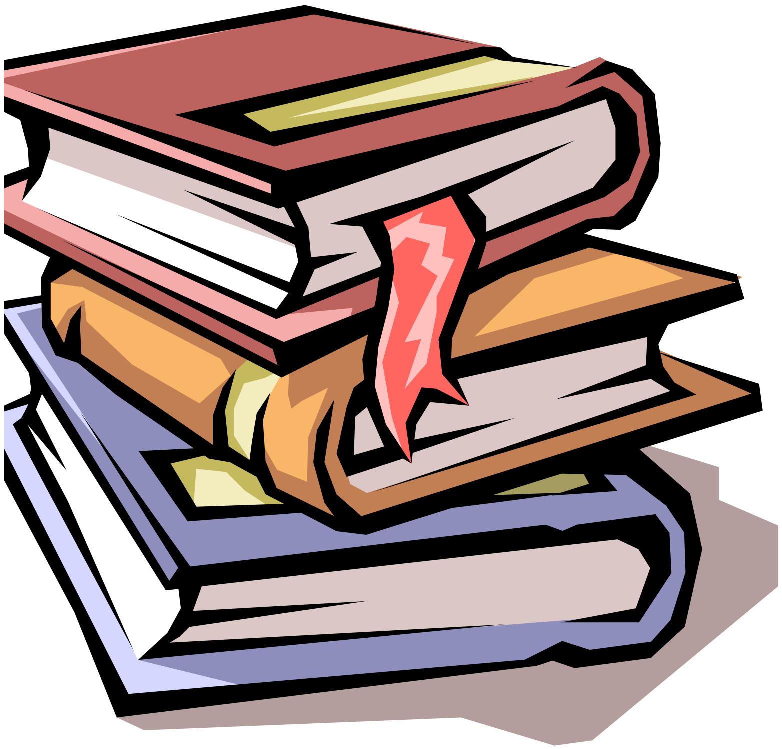
$$x + y + z = A^2; \quad x^2 + y^2 + z^2 = B^4$$

  namely, x = 409, y = 152, and z = 64 (among other solutions).
- Then Euler found four such numbers: 193, 104, 48, and 16.
- Then he found five such numbers: 89, 72, 32, 16, and 16.
- Then he pointed out that he had found a pattern in his solution technique, so that the general problem had been solved.

Back to Undergraduates

- Last spring, I assigned a project in which students had to engage with a primary source or a translation of a primary source.
- One student chose this paper.
- She found six numbers that had the same property. Namely: 97, 112, 64, 64, 64, and 128.
- Their sum is $529 = 23^2$, and the sum of their squares is $50625 = 15^4$.

What I Learned



- Always cite your sources.
- Become friends with your librarian.
- Translating can be fun.
- Diophantine geometry can lead to interesting mathematics, even for undergraduates.

Thank You!